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Poynting vector & poynting Theorem

* **Poynting vector**:- At any point in a uniform plane wave, the cross product of electric field intensity ($\vec{E}$) and magnetic field intensity ($\vec{H}$) gives measure of rate of energy flow per unit surface area at that point.

Mathematically $\vec{S} = \vec{E} \times \vec{H}$

Where $\vec{S}$ is known as poynting vector.

Therefore cross product of electric field intensity $\vec{E}$ and magnetic field intensity $\vec{H}$ at any point in a uniform plane wave, is known as poynting vector. It is denoted by $\vec{S}$.

$$\vec{S} = \vec{E} \times \vec{H} = \frac{\text{Rate of flow of energy}}{\text{Area}}$$

S.I. unit of poynting vector is watt/m^2.

* **Poynting theorem**:- For obtaining the power in uniform plane wave, in 1884 John H. poynting developed a power theorem for electromagnetic field by using Maxwell's equation, which is known as poynting theorem.

Statement of poynting theorem: At any point in a uniform plane wave, the cross product of electric field intensity $\vec{E}$ and magnetic field intensity $\vec{H}$ gives measure of rate of energy flow per unit area at that point

Mathematically, $\vec{S} = \vec{E} \times \vec{H}$ = poynting vector

which is given as

$$\underbrace{\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s}}_{\text{Rate of energy flow}} = -\underbrace{\int_V (\vec{E} \cdot \vec{J})\, dv}_{\text{Power loss}} - \underbrace{\frac{\delta}{\delta t}\int \left(\frac{\epsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv}_{\text{Rate of decrease in EM energy}}$$

It is known as poynting theorem.

Proof of poynting theorem:- poynting theorem can be proved by using Maxwell's equation.

According to poynting theorem

$$\oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} = -\int_V (\vec{E}\cdot\vec{J})\,dv - \frac{\delta}{\delta t}\int\left(\frac{\epsilon E^2}{2}+\frac{\mu H^2}{2}\right)dv \quad \text{--- (1)}$$

We have to prove equation (1).

From Maxwell's third equation,

$$\vec{\nabla}\times\vec{E} = -\frac{\delta\vec{B}}{\delta t} \Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\delta}{\delta}(\mu\vec{H}) \qquad \because \vec{B}=\mu\vec{H}$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\mu.\frac{\delta\vec{H}}{\delta t}$$

Taking dot product both sides with $\vec{H}$

$$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) = -\mu\left(\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}\right) \quad \text{--- (2)}$$

From Maxwell's fourth equation,

$$\vec{\nabla}\times\vec{H} = \vec{J}+\frac{\delta\vec{D}}{\delta t} \Rightarrow \vec{\nabla}\times\vec{H} = \vec{J}+\frac{\delta(\epsilon\vec{E})}{\delta t} \qquad \because \vec{D}=\epsilon\vec{E}$$

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{J}+\epsilon\frac{\delta\vec{E}}{\delta t}$$

Taking dot product both sides with $\vec{E}$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\vec{J}+\epsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right) \quad \text{--- (3)}$$

Now $\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$ --- (4)

$$\left[\because \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) - \vec{B}\cdot(\vec{A}\times\vec{C})\right]$$

Now using equ's (2) and (3) in equ (4), we get

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\mu\left(\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}\right) - \vec{E}\cdot\vec{J} - \epsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right)$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \left[\epsilon\left(\vec{E}\cdot\frac{\delta\vec{E}}{\delta t}\right)+\mu\left(\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}\right)\right] \quad \text{--- (5)}$$

Now $\frac{\delta(\vec{H}\cdot\vec{H})}{\delta t} = \vec{H}\cdot\frac{\delta\vec{H}}{\delta t}+\vec{H}\cdot\frac{\delta\vec{H}}{\delta t}$ $\left(\because \frac{\delta(xy)}{\delta t} = x\frac{\delta y}{\delta t}+y\frac{\delta x}{\delta t}\right)$

$$\Rightarrow \frac{\delta H^2}{\delta t} = 2\,\vec{H}\cdot\frac{\delta \vec{H}}{\delta t} \qquad \because \vec{H}\cdot\vec{H} = H^2$$

$$\Rightarrow \vec{H}\cdot\frac{\delta \vec{H}}{\delta t} = \frac{1}{2}\frac{\delta H^2}{\delta t} \quad \text{———} \quad (6)$$

Similarly
$$\vec{E}\cdot\frac{\delta \vec{E}}{\delta t} = \frac{1}{2}\frac{\delta E^2}{\delta t} \quad \text{———} \quad (7)$$

Using equⁿ (6) and (7) in equⁿ (5), we get

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \left[\varepsilon\cdot\frac{1}{2}\frac{\delta E^2}{\delta t} + \mu\cdot\frac{1}{2}\frac{\delta H^2}{\delta t}\right]$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \frac{\delta}{\delta t}\left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) \quad \text{———} \quad (8)$$

Integrating equⁿ (8) over the volume v, we get

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = -\int_V (\vec{E}\cdot\vec{J})\,dv - \frac{\delta}{\delta t}\int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv \quad \text{———} \quad (9)$$

But from Gauss' divergence theorem

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s}$$

so
$$\boxed{\oint_S (\vec{E}\times\vec{H})\cdot d\vec{s} = -\int_V (\vec{E}\cdot\vec{J})\,dv - \frac{\delta}{\delta t}\int_V \left(\frac{\varepsilon E^2}{2} + \frac{\mu H^2}{2}\right) dv} \quad \text{———} \quad (1)$$

It is poynting theorem, proved.

Poynting vector ($\vec{E}\times\vec{H}$) or poynting theorem in free space:

For free space, there is conduction of electric current and there is no charge in free space.

so in free space, $\vec{J} = 0$, $\rho = 0$

where $\vec{J}$ = current density and ρ = volume charge density.

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and for free space, $\varepsilon \to \varepsilon_0$, $\mu \to \mu_0$

put $\vec{J} = 0$, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ in equ (1) and poynting theorem or poynting vector $(\vec{E} \times \vec{H})$ in free space will become

$$\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\int_V (\vec{E} \cdot 0)\, dv - \frac{\delta}{\delta t} \int_V \left(\frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2} \right) dv$$

$$\Rightarrow \boxed{\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\delta}{\delta t} \int_V \left(\frac{\varepsilon_0 E^2}{2} + \frac{\mu_0 H^2}{2} \right) dv} \quad \text{---} (10)$$

Equ (10) represents poynting vector $(\vec{S} = \vec{E} \times \vec{H})$ or poynting theorem in free space.